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Height probabilities in solid-on-solid models: II

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Abstract. The RSOS-3 model is a sequence of two-dimensional solid-on-solid models with integer heights restricted between 1 and $r-1$ and nearest neighbours differing by 0, 1 or -1. It has an exact solution manifold with four regimes of distinct physical behaviour along which the height probabilities in the bulk of the system can be calculated. Using some results of the previous letter in this series, we calculate the height probabilities for two of the regimes. Appropriate order parameters are defined and their critical exponents calculated.

1. Introduction

In a previous letter (Forrester and Andrews 1986, hereafter referred to as I) we considered some new solvable classes of two-dimensional lattice models. These models are most naturally presented as solid-on-solid (SOS) models. At each site j of the square lattice, we let there be an integer height variable l_j, $1 \leqslant l_j \leqslant r-1$. For a given integer $n (n \geqslant 2)$, we imposed the constraint that nearest-neighbour heights must differ by

$$\begin{aligned} &0, \pm 1, \pm 2, \ldots, \pm(n-1)/2 \qquad (n \text{ odd}) \\ &\pm 1, \pm 3, \ldots, \pm(n-1) \qquad (n \text{ even}). \end{aligned} \tag{1.1}$$

We denoted such sequences of models by RSOS-n.

As noted in I, the RSOS-2 model is the original eight-vertex SOS model of Andrews *et al* (1984). The parametrisation of the weights along the solvable manifold for the RSOS-3 model was given by Kuniba *et al* (1986) and for all other n by Date *et al* (1986a).

The solvable manifolds of each of the RSOS-n models have four distinct regions of physical behaviour (denoted by regimes I-IV). For the RSOS-2 model the free energy and height probabilities have been calculated exactly in each of the regimes. The calculation turned out to be particularly rich, both from the mathematical viewpoint, yielding generalisations of the Rogers-Ramanujan identities (see, e.g., Andrews 1976) and from the physical viewpoint, yielding explicit realisations of allowed critical exponents according to the conformal theories (Belavin *et al* 1984, Friedan *et al* 1984, Huse 1984, Zamolodchikov and Fateev 1985).

Will the RSOS-n models for $n \geqslant 3$ exhibit the same richness? In I we gave the polynomials representing the height probabilities as combinatorial sums for regime III of the RSOS-n models. For $n=3$ we transformed these polynomials to a series involving Gaussian polynomials (this is the first step in Schur's (1917) proof of the Rogers-Ramanujan identities, and also the approach adopted by Andrews *et al* (1984)). The derivation of this result involved the discovery of some new combinatorial identities, and it appears in general that study of the RSOS-n model will again lead to new results in combinatorial analysis.

In this paper we use the polynomials given in I for the RSOS-3 model to compute the critical exponents β_k for regimes II and III. In regime II we find essentially those β_k of regime II in the RSOS-2 model (but with r replaced by $2r$), which can be understood as a consequence of the underlying symmetry of the ground state. In regime III, for a given r, we find β_k of the same arithmetic form as in regime III of the RSOS-2 model, but the numbers of such β_k are different ($2r-5$ for RSOS-3, $r-3$ for RSOS-2). The results again appear interesting from the viewpoint of critical phenomena.

The presentation of this paper assumes some familiarity with our previous works (Andrews *et al* 1984, Forrester and Baxter 1985, Forrester and Andrews 1986).

2. The solvable manifold of the RSOS-3 model

The standard way to proceed in obtaining an exact solution manifold for a two-dimensional interaction-round-a-face lattice model is to solve the star-triangle equations (STE) (Baxter 1982). Let $W(a, b|c, d)$ denote the Boltzmann weight of a face of the lattice; a, b, c, d being the heights at the sites of the SW (south-west), SE, NW and NE corners respectively. Let W' and W'' denote the Boltzmann weights of the same model but with different values of the couplings. Then the STE is

$$\sum_a W(b, c|g, a)\, W'(g, a|f, e)\, W''(a, c|e, d) = \sum_a W(a, d|f, e)\, W'(b, c|a, d)\, W''(g, b|f, a) \tag{2.1}$$

for all allowed height values $b, c, \ldots, g$.

Let us recall some properties of the solution of the STE for the RSOS-2 model (Baxter 1973, Andrews *et al* 1984).

(i) W is parametrised as a Jacobi theta function, which for a given value of the maximum height $r-1$ depends on two variables u and p. The variable u acts like an anisotropy variable. In particular

$$W(a, b|c, d;\, u=0) = \delta_{a,d} \tag{2.2}$$

where δ denotes the Kronecker delta. The variable p is a temperature-like variable. In particular criticality occurs when $p=0$.

(ii) If $W(a, b|c, d) \equiv W(u, p)$, then $W' = W(u', p)$ and $W'' = W(u'', p)$ with $u'-u = u''$.

(iii) For a given value of r, there is a constant λ such that replacing u by $\lambda - u$ rotates each weight by 90°

$$W(a, b|c, d;\, u, p) = \frac{\phi_a\phi_d}{\phi_b\phi_c}\, W(c, a|d, b;\, \lambda-u, p). \tag{2.3}$$

The prefactor on the right-hand side of (2.3) leaves the partition function and height probabilities unchanged.

(iv) W has the diagonal reflection symmetries

$$W(a, b|c, d) = W(a, c|b, d) = W(d, b|c, a) \tag{2.4}$$

and the 'top-bottom' symmetry

$$W(a, b|c, d) = W(r-a, r-b|r-c, r-d). \tag{2.5}$$

Kuniba *et al* (1986) found a solution of (2.1) for the RSOS-3 model with all the above properties, the only difference being that each of the weights are parametrised as products of two theta functions. They conjectured that the RSOS-n model possesses an exact solution manifold on which the Boltzmann weights can be parametrised as linear combinations of $(n-1)$ theta functions. This was subsequently confirmed by Date *et al* (1986a) who properly understood the nature of these solutions as a fusion of the weights for the original RSOS-2 model. It was then noted by Akutsu *et al* (1986) that the fused weights are closely related to (if not precisely) periodic Z-invariant RSOS-2 models (the concept of Z invariance is due to Baxter (1978); see Perk and Wu (1986) for a review). This would have the consequence that the height probabilities for each of the RSOS-n models are precisely the same. We find that this is not in general true, so this feature of the exact solution requires further study.

It is our aim to calculate the height probabilities for the RSOS-3 model. Due to the symmetries (2.4) and (2.5) there are seven different classes of face weights. Following Kuniba *et al* (1986) we label them A_l, B_l, C_l, D_l, P_l, Q_l and R_l (see figure 1).

We noted above that there are four regions of different physical behaviour. If we replace p by $p^{1/2}$ in the usual definition of the θ_1 function (this convention is used by, e.g., Baxter and Andrews (1986), equation (3.1)) then these regions are

$$\begin{array}{lll} \text{regime I} & p<0 & -\pi/2+\lambda<\mathrm{Re}(u)<0 \\ \text{II} & p>0 & -\pi/2+\lambda<\mathrm{Re}(u)<0 \\ \text{III} & p>0 & 0<\mathrm{Re}(u)<\lambda \\ \text{IV} & p<0 & 0<\mathrm{Re}(u)<\lambda \end{array} \tag{2.6}$$

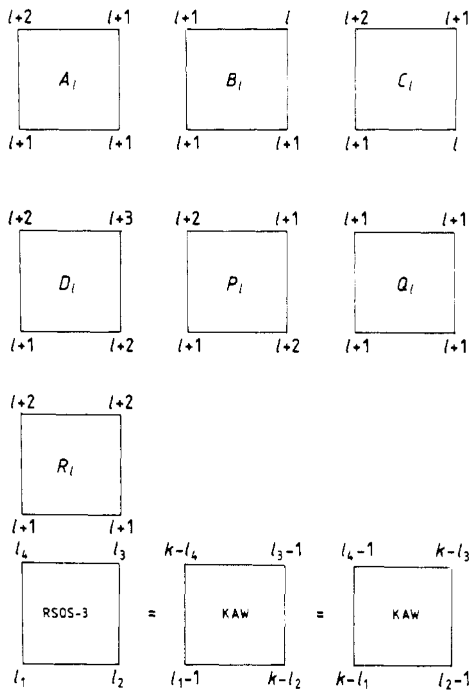

Figure 1. The seven classes of weights for the RSOS-3 model and the mapping from it to the KAW model as presented by Kuniba *et al* (1986).

where

$$\lambda = \pi/2r. \tag{2.7}$$

To calculate the height probabilities it is most convenient to use the conjugate modulus form of the θ_1 function. Define

$$\begin{aligned} p &= \exp(-\varepsilon) \qquad & x &= \exp(-4\pi^2/r\varepsilon) \\ y &= \exp(-4\pi^2/\varepsilon) \qquad & w &= \exp(-4\pi u/\varepsilon) \end{aligned} \tag{2.8}$$

and

$$E(z, y) = \sum_{n=-\infty}^{\infty} (-1)^n y^{n(n-1)/2} z^n. \tag{2.9}$$

In terms of the conjugate modulus representations, the relevant features of the exact solution for purposes of calculating the height probabilities are as follows.

(i) In the limit $\varepsilon \to 0$ the model is in its ground state. For given boundary conditions, this ground state is constant for all $|x| < |w| < 1$ (regimes III, IV) and $1 < |w| < |(x/y^{1/2})|$ (regimes I, II). It changes discontinuously on passing through the endpoints of these regimes.

(ii) The special value

$$W(a, b|\, c, d;\, w = 1) = \delta_{a,d}. \tag{2.10}$$

(iii) The rotation symmetry

$$W(a, b|\, c, d;\, w) = \left(\frac{E(x^a, y)E(x^c, y)}{E(x^b, y)E(x^d, y)}\right)^{1/2} W(c, a|\, d, b;\, x/w). \tag{2.11}$$

(vi) The inversion relation

$$\sum_c W(d, a|\, b, c;\, w)\, W(c, a|\, b, d';\, w^{-1}) = \frac{E(x/w)E(xw)E(x^2w)E(x^2/w)}{(E(x))^2(E(x^2))^2}\, \delta_{d,d'} \tag{2.12}$$

which holds for all allowed heights a, b, d and d'. Here we have abbreviated $E(u, y)$ as $E(u)$.

(v) The functional relation

$$W(a, b|\, c, d;\, w) = \left(\frac{xy}{w^2}\right) \frac{\phi_c \phi_b}{\phi_a \phi_d}\, W(a, b|\, c, d;\, w/y) \tag{2.13}$$

where

$$\phi_a = x^{a^2/2} y^{-a/2}.$$

(vi) The limiting value

$$W(a, b|\, c, d;\, w) \sim w^{|b-c|/2} \delta_{a,d} \qquad \text{as} \qquad x \to 0. \tag{2.14}$$

The notation (2.8) is relevant to regimes II and III when p (and consequently y) is positive. In regimes I and IV properties (i)-(v) still hold but W is not diagonal in the $x \to 0$ limit. For this reason we have not been able to make any progress in calculating the height probabilities for regimes I and IV.

3. Height probabilities for the large but finite lattice

We aim to calculate the probability P_a that a site deep within the lattice has, for given boundary conditions, the value a. This we do using the corner transfer matrix technique (Baxter 1982).

Fixing the boundary heights has the effect of singling out a ground state. In both regime II and III the heights of the ground states do not change along a diagonal. Thus the ground states are specified by the heights along a row. In regime II the ground-state heights along a row are periodic of period $2(r-1)$. A unit cell has the heights in order $1, 2, \ldots, r-2, r-1, r-1, r-2, \ldots, 2, 1$. In regime III the gound-state heights along a row are $l, l+1, l, l+1, \ldots (1 \leqslant l \leqslant r-2)$ and $l, l, l, \ldots (1 \leqslant l \leqslant r-1)$. Thus the ground state in regime II can be singled out by choosing the boundary conditions at the end of a row as $(b, b+1)$ or $(b, b-1)$ while for regime III we have the three types $(b, b+1)$, $(b, b-1)$ and (b, b). Let us denote by $P_a^{\mathrm{X}}(b, c)$ the probability in regime X that the centre site of the lattice has height a, given that the boundary heights are b and c. From the 'top-bottom' symmetry of the weights (2.5) we have

$$P_a^{\mathrm{X}}(b, c) = P_{r-a}^{\mathrm{X}}(r-b, r-c) \tag{3.1}$$

so it suffices to calculate

$$P_a^{\mathrm{II}}(b, b+1) \qquad P_a^{\mathrm{III}}(b, b+1) \qquad P_a^{\mathrm{III}}(b, b) \tag{3.2}$$

for all allowed values of b.

The features of the exact solution (i)-(vi) given in the last section suffice to apply the corner transfer matrix technique and thus obtain the P_a for the large but finite lattice. Following the working of Forrester and Baxter (1985, appendix) step by step we obtain

$$P_a^{\mathrm{II}}(b, c) = x^{(a^2-ra)/2} E(x^a, y)_3 X_m(a, b, c; x^{2(1-r)})/S^{\mathrm{II}}(b, c) \tag{3.3}$$

$$P_a^{\mathrm{III}}(b, c) = E(x^a, y)_3 X_m(a, b, c; x^2)/S^{\mathrm{III}}(b, c) \tag{3.4}$$

where

$$S^{\mathrm{II}}(b, c) = \sum_{a=1}^{r-1} x^{(a^2-ra)/2} E(x^a, y)_3 X_m(a, b, c; x^{2(1-r)}) \tag{3.5}$$

$$S^{\mathrm{III}}(b, c) = \sum_{a=1}^{r-1} E(x^a, y)_3 X_m(a, b, c; x^2) \tag{3.6}$$

and

$${}_3X_m(a, b, c; q) = \sum_{l_2, \ldots, l_m=1}^{r-1} q^{\sum_{k=1}^{r-1} k|l_k - l_{k+2}|/4}. \tag{3.7}$$

In (3.7) $l_1, l_2, \ldots, l_{m+2}$ must satisfy the nearest-neighbour constraint $|l_j - l_{j+1}| = 0$ or 1 and the end heights l_1, l_{m+1}, l_{m+2} are fixed at the values

$$l_1 = a \qquad l_{m+1} = b \qquad l_{m+2} = c. \tag{3.8}$$

In I we transformed the combinatorial sum ${}_3X_m$ to a form suitable for taking the large-m limit. The results (9), (11), (13)-(15) of I can be written as

$${}_3X_m(a, b, c; q) \equiv {}_3X_m(a, b, c)$$

$$= \sum_{\mu=0}^{\infty} q^{(\mu+\tau/2)/2} \begin{bmatrix} m \\ 2(\mu+\tau/2) \end{bmatrix}_{q^{1/2}} {}_2X_{m-2(\mu+\tau/2)}(a, b, c; q) \tag{3.9}$$

where $c=b\pm 1$ and $\tau=0$ for $m-b+a$ even and $\tau=1$ otherwise. The symbol

$$\begin{bmatrix} M \\ N \end{bmatrix}_q$$

denotes the Gaussian polynomial of argument q (see, e.g., Andrews 1976). Here

$${}_2X_m(a,b,c;q)=\sum q^{\sum_{k=1}^{m} k|l_k-l_{k+2}|/4} \tag{3.10}$$

where the sum is the same as in (3.7) except now the nearest-neighbour constraint is $|l_j-l_{j+1}|=1$. From Andrews *et al* (1984, equations (2.3.5) and (2.3.6)) we have the representation

$${}_2X_m(a,b,c;q)=q^{a(a-1)/4}(F_m(a,b,c)-F_m(-a,b,c)) \tag{3.11}$$

where

$$F_m(a,b,c)=\sum_{\lambda=-\infty}^{\infty} q^{r(r-1)\lambda^2+\alpha(a,b,c)\lambda+\beta(a,b,c)}\begin{bmatrix} m \\ (m+a-b)/2-r\lambda \end{bmatrix}_q \tag{3.12}$$

$$\alpha(a,b,c)=r(b+c-1)/2-a(r-1) \tag{3.13}$$

$$\beta(a,b,c)=[bc-a(b+c-1)]/4. \tag{3.14}$$

Expressions (10) and (12) of I for ${}_3X_m(a,b,b)$ can also be written in a form analogous to (3.9), but we prefer the original representations.

4. Regime II

From the discussion at the beginning of § 3 it suffices to calculate $P_a^{\mathrm{II}}(b, b+1)$ $(1\leq b\leq r-2)$ and thus from (3.5) we require the large-m behaviour of ${}_3X_m(a,b,b+1;q^{-1})$ when the argument q^{-1} is greater than 1.

Analogous to Andrews *et al* (1984, equation (2.6.1)) we define

$${}_nx_m(a,b,b+1)=q^{m(m+1)/4}\,{}_nX_m(a,b,b+1;q^{-1}). \tag{4.1}$$

Then, together with the relation

$$\begin{bmatrix} M \\ N \end{bmatrix}_{q^{-1}}=q^{-N(M-N)}\begin{bmatrix} M \\ N \end{bmatrix}_q \tag{4.2}$$

(3.9) becomes

$${}_3x_m(a,b,b+1)=\sum_{\mu=0}^{\infty} q^{(\mu+\tau/2)^2}\begin{bmatrix} m \\ 2\mu+\tau \end{bmatrix}_{q^{1/2}}{}_2x_{m-2\mu-\tau}(a,b,b+1). \tag{4.3}$$

Further, we introduce the function (Andrews *et al* 1984, equation (2.6.52))

$${}_2\hat{x}_m(a,b,b+1)=q^{-(m-b)(m-b+r)/(4r-8)}\,{}_2x_m(a,b,b+1) \tag{4.4}$$

so that (4.3) can be written

$$\begin{aligned}{}_3x_m(a,b,b+1)={}&q^{(m-b)(m-b+r)/(4r-8)}\sum_{\mu=0}^{\infty} q^{[(r-1)/(r-2)](\mu+\tau/2)^2}q^{-(m-b+r/2)(\mu+\tau/2)/(r-2)}\\ &\times\begin{bmatrix} m \\ 2\mu+\tau \end{bmatrix}_{q^{1/2}}{}_2\hat{x}_{m-2\mu-\tau}(a,b,b+1).\end{aligned} \tag{4.5}$$

Write $m = 2(r-1)M + m_0$, $0 \leq m_0 \leq 2(r-1)-1$. Then replacing μ by $\mu + M$ in (4.5) and defining ${}_3\hat{x}_m$ as

$$ {}_3x_m(a, b, b+1) = q^{(m-b)(m-b+r)/(4r-8) - M(m+m_0+r-2b)/(2r-4)}\, {}_3\hat{x}_m(a, b, b+1) \tag{4.6} $$

we have

$$ {}_3\hat{x}_m(a, b, b+1) = \sum_{\mu=-M}^{\infty} q^{[(r-1)/(r-2)](\mu+\tau/2)^2 - (m_0-b+r/2)(\mu+\tau/2)/(r-2)} \times \begin{bmatrix} m \\ 2M+2\mu+\tau \end{bmatrix}_{q^{1/2}} {}_2\hat{x}_{m-2M-2\mu-\tau}(a, b, b+1) \tag{4.7} $$

but

$$ \begin{bmatrix} m \\ 2M+2\mu+\tau \end{bmatrix}_{q^{1/2}} \sim \frac{1}{Q(q^{1/2})} \qquad \text{as } M \to \infty \tag{4.8} $$

and from Andrews *et al* (1984, equation (2.6.53))

$$ {}_2\hat{x}_{m-2M-2\mu-\tau}(a, b, b+1) \sim q^{a(r-a)/(4r-8)}\, \hat{\eta}_{a,(a-b+m-2M-2\mu-\tau)/2}. \tag{4.9} $$

Here

$$ \hat{\eta}_{a,j} = q^{-[(1/2)rj(j+1)-aj]/(r-2)} \eta_{a,j} \tag{4.10} $$

and the $\eta_{a,j}$ are defined by

$$ \Phi_a(z) = \sum_{j=-\infty}^{\infty} \eta_{a,j} z^j \tag{4.11} $$

$$ \Phi_a(z) = \frac{E(q^a, q^r)E(-z, q)(Q(q^r))^3}{E(-z, q^r)E(-q^a z, q^r)(Q(q))^2}. \tag{4.12} $$

In (4.12) the E function is defined by (2.9) and

$$ Q(q) = \prod_{j=1}^{\infty} (1-q^j). \tag{4.13} $$

The $\hat{\eta}_{a,j}$ have the property

$$ \hat{\eta}_{a,j} = \hat{\eta}_{a,j+(r-2)} \tag{4.14} $$

(Andrews *et al* 1984, equation (2.6.45)). Hence

$$ {}_3\hat{x}_m(a, b, b+1) \sim \frac{1}{Q(q^{1/2})} q^{a(r-a)/(4r-8)} \sum_{\mu=-\infty}^{\infty} q^{[(r-1)/(r-2)](\mu+\tau/2)^2} \times q^{-(m_0-b+r/2)(\mu+\tau/2)/(r-2)} \hat{\eta}_{a,(a-b+m_0-2\mu-\tau)/2}. \tag{4.15} $$

In the μ summation we write

$$ \mu = (r-2)\nu + \alpha \qquad \nu = 0, \pm 1, \pm 2, \ldots \qquad \alpha = 0, 1, \ldots, r-3. \tag{4.16} $$

Then, using the periodicity property (4.14) we have

$$ {}_3\hat{x}_m(a, b, b+1) \sim \frac{1}{Q(q^{1/2})} q^{a(r-a)/(4r-8)} \sum_{\alpha=0}^{r-3} q^{[(r-1)/(r-2)](\alpha+\tau/2)^2} \times q^{-(m_0-b+r/2)(\alpha+\tau/2)/(r-2)} \hat{\eta}_{a,(a-b+m_0-2\alpha-\tau)/2} \times E(-q^{(r-1)(r-2)-m_0+b-r/2+(r-1)(2\alpha+\tau)}, q^{2(r-1)(r-2)}). \tag{4.17} $$

Since ${}_3\hat{x}_m$ differs from ${}_3X_m(q^{-1})$ by factors independent of a, substitution of (4.17) into (3.3) evaluates $P_a^{11}(b, b+1)$ in terms of the ground-state variables x and y.

To expand our result about criticality we need to convert back to the original criticality variable p. We require the conjugate modulus identities

$$\theta_1(u, \exp(-\varepsilon)) = \rho(u, \varepsilon)E(\exp(-4\pi u/\varepsilon), \exp(-4\pi^2/\varepsilon)) \qquad (4.18a)$$

$$\theta_4(u, \exp(-\varepsilon)) = \rho(u, \varepsilon)E(-\exp(-4\pi u/\varepsilon), \exp(-4\pi^2/\varepsilon)) \qquad (4.18b)$$

where

$$\theta_1(u, \exp(-\varepsilon)) = -\mathrm{i}\exp(-\varepsilon/8)\sum_{n=-\infty}^{\infty}(-1)^n \times\exp[-\varepsilon n(n-1)/2]\exp[2\mathrm{i}u(n+1/2)] \qquad (4.18c)$$

$$\theta_4(u, \exp(-\varepsilon)) = \sum_{n=-\infty}^{\infty}(-1)^n\exp(-\varepsilon n^2/2)\exp(2\mathrm{i}un) \qquad (4.18d)$$

$$\rho(u, \varepsilon) = (2\pi/\varepsilon)^{1/2}\exp[(2\pi u - 2u^2 - \pi^2/2)/\varepsilon]. \qquad (4.18e)$$

To apply these transformations to the function $\hat{\eta}_{a,j}(\exp[-8\pi^2(r-1)/r\varepsilon])$ we proceed as in Andrews *et al* (1984, equations (3.3.7)-(3.3.15)).

Equation (3.3.2) of Andrews *et al* (1984) can be written

$$\Phi_a(z) = \sum_{j=0}^{r-3} q^{[(1/2)rj(j+1)-aj]/(r-2)} z^j \hat{\eta}_{a,j}(q) E(-q^{r(r-1)/2+rj-a} z^{r-2}, q^{r(r-2)}). \qquad (4.19)$$

With

$$q = x^{2(r-1)} = \exp[-8\pi^2(r-1)/r\varepsilon] \qquad z = \exp[-8\pi(r-1)u/\varepsilon] \qquad (4.20)$$

this identity transforms to

$$F_a(u) = \frac{r}{r-2}\sum_{j=0}^{r-3}\lambda_{a,j}\theta_4(u + [\pi/r(r-2)][\tfrac{1}{2}r(r-1) + rj - a], \exp[-\varepsilon/2(r-1)(r-2)]) \qquad (4.21)$$

where

$$F_a(u) = \frac{\theta_1(\pi a/r, \exp[-\varepsilon/2(r-1)])\theta_4(ur, \exp[-\varepsilon r/2(r-1)])Q^3(\exp[-\varepsilon/2(r-1)])}{\theta_4(u, \exp[-\varepsilon/2(r-1)])\theta_4(u + \pi a/r, \exp[-\varepsilon/2(r-1)])Q^2(\exp[-\varepsilon r/2(r-1)])} \qquad (4.22)$$

$$\lambda_{a,j} = \left(\frac{r-2}{r}\right)^{1/2}\exp\left(-\frac{\pi^2}{3r\varepsilon}(r-1) + \frac{\varepsilon}{r-1}\left(\frac{2r-3}{48}\right) + \frac{4\pi^2(r-1)}{(r-2)}[(a/r)^2 - (a/r)] + \frac{\pi^2(r-1)}{(r-2)\varepsilon}\right)\hat{\eta}_{a,j}(\exp[-8\pi^2(r-1)/r\varepsilon]). \qquad (4.23)$$

The expressions (3.3) and (4.17) can be written in terms of $\lambda_{a,j}$. Using (4.18) we find

$$P_a^{11}(b, b+1) = u_a\left(\sum_{a=1}^{r-1}u_a\right)^{-1} \qquad (4.24)$$

where

$$u_a = \theta_1(\pi a/r, p)\sum_{\alpha=0}^{r-3}\lambda_{a,(a-b+m_0-2\alpha-\tau)/2} \times\theta_4\left(\frac{\pi}{2} - \frac{\pi(m_0 - b + r/2)}{2(r-1)(r-2)} + \frac{\pi(\alpha+\tau/2)}{r-2}, p^{r/[4(r-1)^2(r-2)]}\right). \qquad (4.25)$$

In (4.25) $p=\exp(-\varepsilon)$, $\tau=0$ or 1 according to $a-b+m_0$ being even or odd respectively and $0\leq m_0\leq 2(r-1)-1$.

The variable p is the deviation from criticality parameter, vanishing linearly at the critical point. Critical exponents can thus be calculated by expanding (4.24) in terms of p. To do this for $\lambda_{a,j}$ we note that $F_a(u)$ as given by (4.22) is a π-periodic function and so has a Fourier expansion

$$F_a(u)=\sum_{m=-\infty}^{\infty} f_{a,m}\exp(2\pi i u m). \tag{4.26}$$

Substituting (4.26) into (4.21) and using the definition (4.18*b*) of θ_4, we find that the $\lambda_{a,j}$ are given by

$$\lambda_{a,j}=\frac{1}{r}\sum_{m=0}^{r-3} p^{-m^2/[4(r-1)(r-2)]}\exp\left(-\frac{2\pi i m}{r-2}(j+\tfrac{1}{2}-a/r)\right)f_{a,m}. \tag{4.27}$$

Next, substitute (4.27) into (4.25) and again recall the definition (4.18*b*) of θ_4. The α summation can now be performed and we obtain

$$u_a=\frac{r-2}{r}\theta_1(\pi a/r,p)\sum_{m=0}^{r-3}\sum_{k=-\infty}^{\infty}(-1)^{k\tau}p^{\{r[k^2(r-2)+2km]-m^2\}/8(r-1)^2}$$
$$\times\exp\left(\frac{\pi i(m-k)}{r-1}(b-m_0-\tfrac{1}{2})\right)\exp\left(\frac{-\pi i m a}{r}+\frac{\pi i k}{2}\right)f_{a,m}. \tag{4.28}$$

From (4.26) and (4.22) we have

$$f_{a,m}\sim 2p^{1/16(r-1)}p^{m/4(r-1)}\exp(\pi i m a/r)\sin[\pi a(m+1)/r]. \tag{4.29}$$

The small-p expansion of (4.28) is then obtained from the $k=0$ and $k=-1$ terms ($0\leq m\leq r-3$) and the $k=1$, $m=0$ term (making use of $a-b+m_0+\tau$ being even). Substituting in (4.24) we thus obtain

$$P_a^{\mathrm{II}}(b,b+1)\sim\frac{2}{r}\left[\sin^2\frac{\pi a}{r}+2\sin\frac{\pi a}{r}\sum_{m=1}^{r-2}p^{m[2(r-1)-m]/8(r-1)^2}\right.$$
$$\left.\times\cos\left(\frac{\pi m(b-m_0-\frac{1}{2})}{(r-1)}\right)\sin\left(\frac{\pi a(m+1)}{r}\right)\right]. \tag{4.30}$$

Following Huse (1984), the order parameters are just the spatial Fourier transforms of the P_a

$$\hat{P}_a^{\mathrm{II}}(l)\equiv\frac{1}{2(r-1)}\sum_{m_0=0}^{2(r-1)}P_a^{\mathrm{II}}(b,b+1;m_0)\exp\left(\frac{\pi i l m_0}{r-1}\right). \tag{4.31}$$

From (4.30) we have

$$\hat{P}_a^{\mathrm{II}}(l)\sim\frac{2}{r}p^{\beta_l}\sin\left(\frac{\pi a}{r}\right)\sin\left(\frac{\pi(l+1)a}{r}\right)\exp\left(\frac{\pi i l}{r-1}(b-l-\tfrac{1}{2})\right) \tag{4.32}$$

with exponent

$$\beta_l=\frac{l[2(r-1)-l]}{8(r-1)^2}\qquad 1\leq l\leq 2r-3\qquad l\neq r-1. \tag{4.33}$$

For $l=r-1$ the order parameter (4.31) vanishes identically.

5. Regime III

Firstly we remark that regime III for the RSOS-n models in general has been discussed by Date *et al* (1986b), where implicit expressions for the P_a are given.

Here we will obtain explicit expressions for the P_a in regime III of the RSOS-3 model. From (3.4) we note that the argument of ${}_3X_m$ is less than one. Thus from (3.9) we have immediately for large m

$$ {}_3X_m(a, b, b+1) \sim \left(\sum_{\mu=0}^{\infty} \frac{q^{(\mu+\tau/2)}}{(q^{1/2}, q^{1/2})_{2\mu+\tau}} \right) {}_2X_\infty(a, b, b+1) \tag{5.1} $$

where

$$ {}_2X_\infty(a, b, b+1) = \frac{1}{Q(q)} q^{b(b+1)/4} [q^{-ab/2} E(-q^{(r-a)(r-1)+rb}, q^{2r(r-1)}) - q^{ab/2} E(-q^{(r+a)(r-1)+rb}, q^{2r(r-1)})]. \tag{5.2} $$

The case ${}_3X_m(a, b, b)$ requires more work. From equations (10) and (12) of I, we have

$$ {}_3X_m(a, b, b) \sim \delta_{a,b} + q^{1/4+a(a-1)/4}(h(a, b+1, b) - h(-a, b+1, b) + h(a, b-1, b) - h(-a, b-1, b)) \tag{5.3} $$

where

$$ h(a, b', b) = \sum_{\lambda=-\infty}^{\infty} q^{r(r-1)\lambda^2 + \alpha(a, b', b)\lambda + \beta(a, b', b)} S_\lambda(|a-b'|) \tag{5.4} $$

$$ \alpha(a, b', b) = \mathrm{sgn}(b'-a)[r(b'+b-1)/2 - a(r-1)] \tag{5.5} $$

$$ \beta(a, b', b) = b'b/4 - a(b'+b-1)/4 \tag{5.6} $$

$$ S_\lambda(d) = q^{d/4} \sum_{\mu=0}^{\infty} q^{\mu/2} \frac{1}{(q^{1/2}, q^{1/2})_{2\mu+d+1}} \begin{bmatrix} 2\mu+d \\ \mu - r\lambda \end{bmatrix}_q . \tag{5.7} $$

We have used the notation

$$ \mathrm{sgn}(t) = 1 \qquad \text{for } t \geqslant 0 \text{ and } -1 \text{ otherwise} \tag{5.8} $$

$$ (a; q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1}). \tag{5.9} $$

In the appendix we establish the summation formula

$$ S_0(d) = q^{d/4} \chi \sum_{n=0}^{\infty} (-1)^n q^{n^2/2+(d+1)n/2} \qquad d \geqslant 0 \tag{5.10} $$

where

$$ \chi = (-q^{1/2}, q^{1/2})_\infty / (q^{1/2}, q^{1/2})_\infty = \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} \right)^{-1} . \tag{5.11} $$

We can express $S_\lambda(d)$ in terms of $S_0(d)$. Firstly we note $S_0(-d)$ is simply related to $S_0(d)$. Suppose d is even. Then by replacing $\mu + d/2$ in (5.7) we have

$$ S_0(-d) = \sum_{\mu=d/2}^{\infty} \frac{q^{(\mu+d/2)/2}}{(q^{1/2}, q^{1/2})_{2\mu+1}} \begin{bmatrix} 2\mu \\ \mu + d/2 \end{bmatrix}_q . \tag{5.12} $$

If we use the symmetry of the Gaussian polynomial

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} M \\ M-N \end{bmatrix} \tag{5.13}$$

in (5.12), we deduce

$$S_0(-d) = q^{d/4} S_0(d). \tag{5.14}$$

When d is odd we obtain the same relation.

Now consider $S_\lambda(d)$. For $\lambda \geq 0$ we can begin the μ summation in (5.7) at $r\lambda$, since the other terms vanish. Now replace μ by $\mu + r\lambda$ so the μ summation is again from zero. This gives

$$S_\lambda(d) = S_0(d + 2r\lambda). \tag{5.15}$$

A similar simple agrument shows (5.15) to be true for all integers λ provided $0 \leq d \leq 2r-1$.

Hence from (5.10), (5.14) and (5.15)

$$S_\lambda(d) = q^{(d+2r\lambda)/4} \chi \sum_{n=0}^{\infty} (-1)^n q^{n^2/2+(d+1+2r\lambda)n/2} \qquad \lambda \geq 0 \tag{5.16}$$

$$S_\lambda(d) = q^{-(d+2r\lambda)/4} \chi \sum_{n=0}^{\infty} (-1)^n q^{n^2/2+(-d+1-2r\lambda)n/2} \qquad \lambda < 0 \qquad 0 \leq d \leq 2r-1. \tag{5.17}$$

Define

$$H_+(a) = h_+(a, b+1, b) + h_+(a, b-1, b) \tag{5.18}$$

where h_+ is defined as in (5.4) except that the λ summation is from 0 to ∞. Suppose $a > b+1$, and substitute in the results (5.16) and (5.17). Then replacing n by $n-1$ in $h_+(a, b-1, b)$ shows that both summands are identical but opposite in sign, so the only term surviving is $n = 0$ in $h_+(a, b+1, b)$ (since after replacing n by $n-1$ the summation in $h_+(a, b-1, b)$ is from 1 to ∞). Hence

$$H_+(a) = \chi q^{(b^2-2ab+a-1)/4} \sum_{\lambda=0}^{\infty} q^{r(r-1)\lambda^2 - [r(b-a)+a-r/2]\lambda} \qquad a \geq b+1. \tag{5.19}$$

Similarly define

$$H_-(a) = h_-(a, b+1, b) + h_-(a, b-1, b) \tag{5.20}$$

where h_- is defined as in (5.4) but with λ summation from $-\infty$ to -1. Again substitute in the results (5.16) and (5.17). This time replacing n by $n-1$ in $h_-(a, b+1, b)$ shows both summands are equal but opposite in sign, so the only surviving term is $n = 0$ in $h_-(a, b-1, b)$. Hence

$$H_-(a) = \chi q^{(b^2-2ab+a-1)/4} \sum_{\lambda=-\infty}^{-1} q^{r(r-1)\lambda^2 - [r(b-a)+a-r/2]\lambda} \qquad a \geq b+1. \tag{5.21}$$

Adding (5.20) and (5.21) together we have

$$\begin{aligned} h(a, b+1, b) + h(a, b-1, b) &\equiv H_+(a) + H_-(a) \\ &= \chi q^{(b^2-2ab+a-1)/4} E(q^{(r-a)(r-1)+r(b-1/2)}, q^{2r(r-1)}). \end{aligned} \tag{5.22}$$

The cases $a<b+1$ are handled in the same way, with (5.22) again holding. The only exception is the case $a=b$, when the left-hand side of (5.22) is to be replaced by

$$q^{-(a^2-a+1)/4}+h(a, a+1, a)+h(a, a-1, a). \tag{5.23}$$

Using these results, we have from (5.3) the large-m evaluation

$$\begin{aligned} {}_3X_m(a, b, b) \sim \chi q^{a(a-1)/4+b^2/4}[&q^{-a(b-1/2)/2}E(q^{(r-a)(r-1)+r(b-1/2)}, q^{2r(r-1)}) \\ &-q^{a(b-1/2)/2}E(q^{(r+a)(r-1)+r(b-1/2)}, q^{2r(r-1)})]. \end{aligned} \tag{5.24}$$

Substituting (5.2) and (5.24) into (3.4) evaluates $P_a^{\mathrm{III}}(b, b+1)$ and $P_a^{\mathrm{III}}(b, b)$ respectively. From the work of Andrews *et al* (1984), we expect that these results can be further simplified via identities expressing the denominators as single theta functions. Indeed, the necessary result is contained within the proof of theorem (3.2.1) of Andrews *et al* (1984). We have the following.

Theorem 5.1. Let x, y be real numbers such that $|x|<1$ and

$$y^{m/2}=-\varepsilon x^r \qquad \varepsilon=\pm 1 \tag{5.25}$$

where m, r are positive integers and $1\leqslant m<2r$; then for all complex numbers z

$$\begin{aligned} \sum_{a=-(r-1)}^{r-1} x^{(1/2)a(a-1)}z^aE(x^a, y)E(\varepsilon^{m-1}x^{(2r-m)(r+a)}z^{2r}, x^{2r(2r-m)}) \\ = E(-z, x)E(z^{-1}, y/x). \end{aligned} \tag{5.26}$$

Choosing $m=2$, $\varepsilon=-1$, $y=x^r$ and $z=x^{-b}$ gives the summation theorem for the denominator of $P_a^{\mathrm{III}}(b, b+1)$. To see that note that

$$E(-x^{2(r-1)(r+a)-2rb}, x^{4r(r-1)})=E(-x^{2(r-1)(r-a)+2rb}, x^{4r(r-1)}) \tag{5.27}$$

and

$$E(x^{-a}, y)=-x^{-a}E(x^a, y). \tag{5.28}$$

The summation theorem then follows by grouping the $(-a, a)$ terms together. Similarly, the choice $m=2$, $\varepsilon=-1$, $y=x^r$ and $z=x^{-(b-1/2)}$ gives the required summation formula for the denominator of $P_a^{\mathrm{III}}(b, b)$.

After using the conjugate modulus identities (4.18) our evaluations of the P_a^{III} become

$$P_a^{\mathrm{III}}(b, c)=\frac{\theta_3(\pi a/2r-\pi d/(2r-2), p^{r/4(r-1)})-\theta_3(\pi a/2r+\pi d/(2r-2), p^{r/4(r-1)})}{R_a\theta_4(\pi d, p^r)\theta_1(\pi d/(r-1), p^{r(r-1)})} \tag{5.29}$$

where

$$d=(b+c-1)/2 \qquad c=b, b+1 \qquad R_a=2r/\theta_1(\pi a/r, p) \tag{5.30}$$

and

$$\theta_4(u+\pi/2, p)=\theta_3(u, p). \tag{5.31}$$

Note that for $c=b+1$ this is, up to a factor of $\frac{1}{2}$, identical to $P_a(b, b+1)$ for regime III of the RSOS-2 model (Andrews *et al* 1984, equation (3.3.18*c*)).

From (5.29), for small p

$$P_a^{\mathrm{III}}(b,c) \sim \frac{\sin(\pi a/r)}{r\sin[\pi d/(r-1)]} \sum_{n=1}^{2r-3} p^{(n^2-1)/8(r-1)} \sin(n\pi a/r)\sin[n\pi d/(r-1)][1+\mathrm{O}(p)]. \tag{5.32}$$

There are $2r-3$ different phases (up to translations) which can be labelled by $D=2d$, $(D=1,2,\ldots,2r-3)$. Following Huse (1984) we take the order parameters in regime III as

$$R_a(k)=\frac{1}{2(r-1)}\sum_{D=1}^{2(r-1)} P_a^{\mathrm{III}}(D)\sin[\pi D/2(r-1)]\sin[\pi D(k+1)/2(r-1)] \tag{5.33}$$

for integers $1\leqslant k\leqslant 2(r-2)$. From (5.32) we have as $p\to 0$

$$R_a(k)=\frac{p^{\beta_k}}{2r}\sin(\pi a/r)\sin[\pi a(k+1)/r] \tag{5.34}$$

where the critical exponents are

$$\beta_k=\frac{(k+1)^2-1}{8(r-1)} \qquad k=1,2,\ldots,2(r-2) \qquad k\neq r-1. \tag{5.35}$$

For $k=r-1$ the order parameter (5.33) vanishes identically.

6. The limit $r\to\infty$ in regime III

Consider the parametrisation in terms of the conjugate modulus variables (2.8) for the weights in regime III of the RSOS-3 model. We noted in I that in the limit r, a, $b\to\infty$, $(a-b)=$ constant these weights reduce to those of the three-state vertex model as given by Sogo *et al* (1983). When viewed as a solid-on-solid model, the three-state vertex model is of interest for the richness of its phase diagram (Glaus 1986, Truong and den Nijs 1986) which is related to that of the spin-1 quantum chain.

From (5.1) and (5.24) we can write down the expressions for height probabilities in the bulk along the exact solution manifold. We have

$$P_a(\varepsilon)=x^{(1/2)(a^2-\varepsilon a)}/E(-x^{(1-\varepsilon)/2},x). \tag{6.1}$$

Here ε is a phase label, equal to zero when the phase has an underlying ground state with heights along the centre row 0000..., and equal to one when the ground state has heights along the centre row 01010.... In the isotropic case ($w=x^{1/2}$) when $A=B$, $C=D$ (recall figure 1), the parameter x is given in terms of the Boltzmann weight A by

$$A\equiv\exp(-J/k_\mathrm{B}T)=x^{1/4}/(1+x^{1/2}). \tag{6.2}$$

In the limit $x\to 1^-$ the model becomes critical. From (6.2) this corresponds to

$$J/k_\mathrm{B}T\to\log 2 \tag{6.3}$$

which marks the onset of a rough phase in which the P_a are zero for each a. We are interested in the singular behaviour of P_a and

$$\langle h^2\rangle\equiv\sum_{a=-\infty}^{\infty} a^2 P_a \tag{6.4}$$

as $x\to 1^-$.

Defining the deviation from criticality parameter by

$$t=(T-T_c)/T_c \tag{6.5}$$

with T_c specified by (6.3) we have from (6.2)

$$\tfrac{1}{4}\log x \sim -(2\log 2)^{1/2}|t|^{1/2}.$$

Using the formulae

$$E(-x^{(1-\varepsilon)/2},x)\sim\left(\frac{2\pi}{-\log x}\right)^{1/2} \tag{6.6}$$

$$\sum_{a=-\infty}^{\infty} a^2 x^{(1/2)(a^2-\varepsilon a)} \sim \frac{1}{2\pi}\left(\frac{2\pi}{-\log x}\right)^{3/2} \tag{6.7}$$

we have

$$P_a \sim (-\log x/2\pi)^{1/2} = [2(2\log 2)^{1/2}/\pi]^{1/2}|t|^{1/4} \tag{6.8}$$

$$\langle h^2\rangle \sim (-\log x)^{-1} = [4(2\log 2)^{1/2}]^{-1}|t|^{-1/2}. \tag{6.9}$$

These results are independent of the phase, and are identical to those obtained for the BCSOS model (Forrester 1986, equations (19)-(22) after some corrections to the working therein) except for a factor $\frac{1}{2}$ in (6.8).

7. Comparison of the critical exponents with those of the RSOS-2 model and conformal field theories

From the results of Andrews *et al* (1984), Huse (1984) calculated the critical exponents for the RSOS-2 model. In regime II, where there are $r-2$ phases with $(r-2)\times 1$ symmetry, the critical exponents are

$$\beta_k=\frac{k(r-2-k)}{2(r-2)^2} \qquad k=1,2,\ldots,r-3 \tag{7.1}$$

$$2-\alpha=r/(r-2). \tag{7.2}$$

In regime II of the RSOS-3 model, there are $2(r-1)$ phases with $2(r-1)\times 1$ symmetry, and we calculated (4.33)

$$\beta_k=\frac{k[2(r-1)-k]}{8(r-1)^2} \qquad k=1,2,\ldots,2r-3 \qquad k\neq r-1. \tag{7.3}$$

For $k=r-1$, the order parameter vanishes identically, which seems to be due to an extra symmetry of the ground state—the height at site l of the unit cell is unchanged by the rearrangement $l\to 2r-1-l$. From the inversion relation method (see, e.g., Baxter 1982) we can calculate the free energy in regime II from the inversion relation (2.12) and thus obtain

$$2-\alpha=r/(r-1). \tag{7.4}$$

Replacing r by $2r$ in (7.1) and (7.2) we can go from the RSOS-2 results to the RSOS-3 results (7.3) and (7.4) (apart from the vanishing order parameter). This makes good sense, considering the type of ordered phases ($(r-2)\times 1$ in the former; $2(r-1)\times 1$ in the latter), and the further symmetry of the unit cell in the RSOS-3 case. The exponent

$\eta_l = 2\beta_l/(2-\alpha)$ was identified by Zamolodchikov and Fateev (1985) as that of a conformal field theory with Z_{r-2} symmetry and central charge

$$c = 2(r-3)/r. \tag{7.5}$$

In regime III of the RSOS-2 model the critical exponents are (Huse 1984)

$$\beta_k = \frac{(k+1)^2-1}{8(r-1)} \qquad k = 1, 2, \ldots, r-3 \tag{7.6}$$

$$2-\alpha = r/2 \tag{7.7}$$

while in regime III of the RSOS-3 model we have (5.35)

$$\beta_k = \frac{(k+1)^2-1}{8(r-1)} \qquad k = 1, 2, \ldots, 2(r-2) \qquad k \neq r-1 \tag{7.8}$$

$$2-\alpha = r$$

(the latter result can be deduced from the inversion relation (2.12)). For $k = r-1$ the order parameter vanishes identically. The exponents $\eta_l = 2\beta_l/(2-\alpha)$ for regime III of the RSOS-2 model have been identified by Huse (1984) as those of the conformal field theory with central charge

$$c = 1 - 6/r(r-1) \tag{7.9}$$

and no special symmetries linking the phases (the exponents for such a field theory were given by Friedan *et al* (1984)).

In regime III of the RSOS-3 model, it has been stated by Date *et al* (1986b) that the critical exponents η_l are that of a conformal field theory with central charge

$$c = \tfrac{3}{2} - 3/r(r-1). \tag{7.10}$$

Note that for $r=3$ we have $c=1$ which is the central charge of the eight-vertex model. That the $r=3$ case of the RSOS-3 model is a special case of the eight-vertex model can easily be seen directly from the definition of the model.

Appendix

Throughout we will adopt the notation (5.9) as well as

$$(a;q)_n = (a)_n \tag{A1}$$

and we will introduce the q-hypergeometric series

$${}_n\Phi_s\begin{pmatrix}\alpha_1, \ldots, \alpha_n; q, t\\ \beta_1, \ldots, \beta_s\end{pmatrix} = \sum_{j=0}^{\infty} \frac{(\alpha_1, \ldots, \alpha_n; q)_j t^j}{(q, \beta_1, \ldots, \beta_s; q)_j} \tag{A2}$$

where

$$(A_1, A_2, \ldots, A_r; q)_n = (A_1)_n (A_2)_n \ldots (A_r)_n. \tag{A3}$$

Here we prove the identity

$$\sum_{\mu=0}^{\infty} \frac{q^{\mu/2}}{(q^{1/2}; q^{1/2})_{2\mu+d+1}} \begin{bmatrix} 2\mu+d \\ \mu \end{bmatrix}_q = \chi \sum_{n=0}^{\infty} (-1)^n q^{[n^2+(d+1)n]/2} \qquad d \geq 0 \tag{A4}$$

where

$$\chi = (-q^{1/2}; q^{1/2})_\infty / (q^{1/2}; q^{1/2})_\infty. \tag{A5}$$

Theorem A1. For complex numbers z and $|q|<1$

$$\sum_{\mu=0}^{\infty} \frac{q^{\mu}}{(q^2;q^2)_{\mu}} \frac{(z^2q^{2\mu+2};q^2)_{\infty}(zq^{2\mu+2})_{\infty}}{(z^2q^{4\mu+2};q^2)_{\infty}} = (-q)_{\infty} \sum_{m=0}^{\infty} (-1)^m q^{m^2+m} z^m. \tag{A6}$$

The identity (A4) follows from (A6) by choosing $z=q^d$, $d\geqslant 0$, replacing q by $q^{1/2}$ and noting

$$(aq^n;q)_{\infty} = (a;q)_{\infty}/(a;q)_n \qquad n\geqslant 0. \tag{A7}$$

Proof. Denote the left-hand side of (A6) by $S(z)$. Then using the relation (A7) we can write

$$S(z) = (zq^2)_{\infty} \sum_{\mu=0}^{\infty} \frac{(z^2q^2;q^2)_{2\mu} q^{\mu}}{(q^2;q^2)_{\mu}(z^2q^2;q^2)_{\mu}(zq^2)_{2\mu}} \tag{A8}$$

which can easily be identified as

$$S(z) = (zq^2)_{\infty}\, {}_3\Phi_2\left(\begin{matrix} -zq, -zq^2, zq; q^2, q \\ z^2q^2, zq^3 \end{matrix}\right). \tag{A9}$$

By an identity of Sears (1951, equation (10.1)) with $a=-zq$, $b=-zq^2$, $c=zq$, $e=zq^3$, $f=z^2q^2$) we then have

$$S(z) = \frac{(zq^2)_{\infty}(zq;q^2)_{\infty}(zq^2;q^2)_{\infty}}{(z^2q^2;q^2)_{\infty}(q;q^2)_{\infty}}\, {}_3\Phi_2\left(\begin{matrix} -q^2, -q, zq; q^2, zq \\ zq^3, zq^2 \end{matrix}\right). \tag{A10}$$

Using the definition (A2) and some simple identities this becomes

$$S(z) = \frac{(zq^2)_{\infty}(zq)_{\infty}}{(z^2q^2;q^2)_{\infty}(q;q^2)_{\infty}} S_1(z) \tag{A11}$$

where

$$S_1(z) = \sum_{n=0}^{\infty} \frac{(-q)_{2n}(zq;q^2)_n (zq)^n}{(zq^2)_{2n}(q^2;q^2)_n}. \tag{A12}$$

By lemma 1 of Andrews (1966), with $k=2$, $a=zq$, $b=-q$, $c=zq^2$ and $t=zq$, we have

$$S_1(z) = \frac{(-q)_{\infty}(z^2q^2;q^2)_{\infty}}{(zq^2)_{\infty}(zq;q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{(-zq)_m(zq;q^2)_m(-q)^m}{(q)_m(z^2q^2;q^2)_m}. \tag{A13}$$

Substituting this in (A11) gives after some straightforward manipulation

$$S(z) = \frac{(zq^2;q^2)_{\infty}(-q)_{\infty}}{(q;q^2)_{\infty}}\, {}_2\Phi_1\left(\begin{matrix} z^{1/2}q^{1/2}, -z^{1/2}q^{1/2}; q, -q \\ zq \end{matrix}\right). \tag{A14}$$

However, by the q analogue of Whipple's theorem (see Slater (1966), equation (3.4.1.5) with d and $g\to\infty$, $e=-f=(zq)^{1/2}$, $c=q$ and $a=zq$), the ${}_2\Phi_1$ function in (A13) is given by

$${}_2\Phi_1 = \frac{(z^{1/2}q^{3/2})_{\infty}(-z^{1/2}q^{3/2})_{\infty}}{(zq^2)_{\infty}(-q)_{\infty}} \sum_{m=0}^{\infty} (-1)^m q^{m^2+m} z^m. \tag{A15}$$

Substituting (A15) in (A14) we obtain, after some simple manipulation, the right-hand side of (A6).

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